25 February 2002

PHYSICS LETTERS A

Physics Letters A 294 (2002) 234–238

www.elsevier.com/locate/pla

Analysis of data from periodically forced stochastic processes

Janez Gradišek [a,*], Rudolf Friedrich [b], Edvard Govekar [a], Igor Grabec [a]

[a] *Faculty of Mechanical Engineering, University of Ljubljana, SI-1000 Ljubljana, Slovenia*
[b] *Institute for Theoretical Physics, Westfälische Wilhelms University, D-48149 Münster, Germany*

Received 3 December 2001; accepted 23 January 2002

Communicated by J. Flouquet

Abstract

A method for analysis of periodically forced stochastic processes is presented. The method enables extraction of the deterministic and random components of process dynamics from measured data, provided the forcing frequency is known and the data is sampled stroboscopically. The method is illustrated by three examples employing synthetic and experimental data. © 2002 Elsevier Science B.V. All rights reserved.

PACS: 05.10.Gg; 05.45.-a; 05.40.Ca

Keywords: Stochastic process; Periodic forcing; Time series analysis

1. Introduction

The main goal of data analysis is to extract information about a process from the data it generates. In recent decades, non-periodic data have been analysed intensively. For non-periodic data generated by deterministic chaotic processes, various analytical methods have been introduced which allow only for a small amount of measurement noise in the data [1]. As such, these methods are not well-suited to analysis of data stemming from stochastic processes of which noise is an integral part. Therefore, other methods are required to extract information from stochastic data.

For a wide class of stochastic processes there exists a general method for non-parametric estimation of the deterministic and random components of process dynamics directly from data [2]. Processes apt for analysis by this method can be described as

$$\frac{d}{dt}\mathbf{X}(t) = \mathbf{h}\big(\mathbf{X}(t)\big) + \mathbf{g}\big(\mathbf{X}(t)\big)\boldsymbol{\Gamma}(t). \tag{1}$$

Here, $\mathbf{X}(t)$ denotes a d-dimensional stochastic variable describing the process state. Its evolution in time is governed by the sum of deterministic and random terms, $\mathbf{h}$ and $\mathbf{g}\boldsymbol{\Gamma}$, where $\boldsymbol{\Gamma}$ and $\mathbf{g}$ denote the uncorrelated Gaussian distributed noise and the matrix of noise amplitudes, respectively. No restricting assumptions for $\mathbf{h}$ and $\mathbf{g}$ are necessary. Based on the method from Ref. [2], various possibilities for further analysis of such stochastic processes have been proposed [3,4].

This Letter extends the above method and its application to the class of periodically forced stochastic processes for which the deterministic term $\mathbf{h}$ in Eq. (1) is explicitly time-dependent, and can be expressed as

* Corresponding author.
E-mail addresses: janez.gradisek@fs.uni-lj.si (J. Gradišek), fiddir@uni-muenster.de (R. Friedrich).

PII: S0375-9601(02)00060-9

the sum of a general and a periodic term:

$$\mathbf{h}\big(\mathbf{X}(t), t\big) = \mathbf{h}_n\big(\mathbf{X}(t)\big) + \mathbf{h}_p(t), \tag{2}$$

where $\mathbf{h}_p(t+T) = \mathbf{h}_p(t)$. We show how the deterministic term $\mathbf{h}$ can be estimated from data, provided that the forcing period T is known and the data is sampled stroboscopically, i.e., exactly k times per forcing period. The method is illustrated by three examples: two examples employ synthetic data from models often applied to biomedical and engineering processes, whereas for the third example, experimental data from interrupted cutting is used.

2. Method

Stochastic processes modeled by Eq. (1) possess the Markovian property. This means that the process state $\mathbf{X}$ at time t depends only on the process state at the preceding time $t-\tau$. The probability that the trajectory of process states visits location $\mathbf{x}_{n+1}$ in state space at time $t+\tau$, given that it visits $\mathbf{x}_n$ at time t, is thus described by the conditional probability density $p(\mathbf{x}_{n+1}, t+\tau|\mathbf{x}_n, t)$ which is independent of the trajectory's path prior to time t. It follows from Eq. (1) that every time t_i the trajectory visits location $\mathbf{x}$, its location at time $t_i+\tau$ is determined by the sum of the deterministic and random functions, $\mathbf{h}(\mathbf{x}, t_i)$ and $\mathbf{g}(\mathbf{x})\boldsymbol{\Gamma}(t_i)$. For fixed $\mathbf{x}$, $\mathbf{h}$ depends periodically on time (Eq. (2)), $\mathbf{g}$ is constant, and $\boldsymbol{\Gamma}(t_i)$ is Gaussian distributed white noise. Adopting Itô's definitions for stochastic integrals, $\mathbf{h}$ and $\mathbf{g}$ can be estimated from the trajectory's path using conditional moments [5]. $\mathbf{g}$ can be estimated as [2]

$$\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^\dagger = \lim_{\tau\to 0}\frac{1}{\tau}\big\langle\big(\mathbf{X}(t_i+\tau)-\mathbf{x}\big)\big(\mathbf{X}(t_i+\tau)-\mathbf{x}\big)^\dagger\big\rangle_{\mathbf{X}(t_i)=\mathbf{x}}. \tag{3}$$

Due to the periodic dependence of $\mathbf{h}$, its estimation requires access to the forcing period T and stroboscopically sampled data. For data sampled exactly k times per forcing period, one can estimate $\mathbf{h}$ using

$$\mathbf{h}(\mathbf{x}, j\Delta t) = \lim_{\tau\to 0}\frac{1}{\tau}\big\langle\mathbf{X}(t_i+\tau)-\mathbf{x}\big\rangle_{\mathbf{X}(t_i)=\mathbf{x}}, \tag{4}$$

where Δt denotes sampling time, $j\Delta t = t_i \bmod T$, and $j = 0, 1, \ldots, k-1$. In other words, to estimate $\mathbf{h}(\mathbf{x})$ at a particular phase angle of forcing, only the points $\mathbf{X}(t_i) = \mathbf{x}$ at that particular phase angle can contribute to the conditional average in Eq. (4).

Using Eqs. (3) and (4), the deterministic and random components of process dynamics can therefore be estimated for every point $\mathbf{x}$ in state space, provided that the point is visited statistically often by the process trajectory $\mathbf{X}(t)$. Because of the limits $\tau\to 0$, it should be verified whether the estimated $\mathbf{h}$ and $\mathbf{g}$ converge as the time step τ is decreased. Once estimates have been obtained, they can be further analysed to extract the properties of both the deterministic flow and noise of the process [3,4].

3. Examples

The method is illustrated below using synthetic and experimental data.

3.1. Stochastic resonance

Stochastic resonance is an important phenomenon found in physical and biomedical sciences (see Ref. [6] and references therein). The mechanism of stochastic resonance can be explained by the motion of a damped particle in a symmetrical double-well potential subjected to randomly fluctuating forces which cause occasional transitions of the particle between the two wells. If the particle is also forced periodically with a forcing amplitude too weak to cause transitions, the noise-induced transitions can become synchronized with the periodic forcing. The motion of the particle can be described as

$$\dot{X}(t) = aX(t) - bX(t)^3 + A\cos(2\pi\nu t) + g\Gamma(t). \tag{5}$$

In our case the parameters were $a = 64$, $b = 4$, $A = 64$, $\nu = 0.5$ Hz, and $g = 14$. The deterministic term h was estimated from a trajectory $X(t)$ generated using Eq. (5). Ten estimates of h within the forcing period are shown in Fig. 1. The estimated curves confirm the sinusoidal dependence of h on time, and agree well with the corresponding theoretical curves.

3.2. Duffing oscillator

A Duffing oscillator is an oscillator with a nonlinear restoring force. It is used, for example, to model

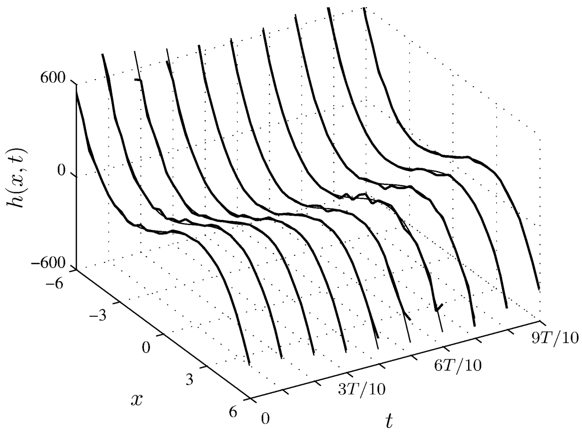

Fig. 1. Estimated deterministic term $h(x,t)$ (thick lines) and its theoretical dependence on state space location x and time t (thin lines).

sinusoidally forced structures undergoing large elastic deflections [7]. The motion of the oscillating mass is governed by

$$\begin{aligned}\dot{X}_1 &= X_2,\\ \dot{X}_2 &= -\epsilon X_2 - \delta X_1 - \mu X_1^3\\ &\quad + A\cos(2\pi\nu t) + g\Gamma(t). \qquad (6)\end{aligned}$$

Parameters $\epsilon = 1$, $\delta = -10$, $\mu = 100$, $A = 0.9$, and $\nu = 0.55$ Hz were chosen so as to ensure a chaotic regime of the deterministic oscillator. Noise amplitude was set to $g = 0.6$. The deterministic term **h** was estimated from a two-dimensional trajectory shown in Fig. 2(a) with its stroboscope section superimposed. The estimated **h** was subsequently employed to reconstruct the deterministic trajectory of the process (Fig. 2(b)) which would be observed if the oscillator was not subjected to noise. Despite the marked differences between the stochastic and deterministic attractors, the reconstructed attractor captures the main features of the two-well Duffing attractor. However, the stroboscope section of the attractor also shows that fine, fractal-like details of the original deterministic attractor could not be completely reproduced.

3.3. Interrupted cutting—milling

In mechanical engineering, cutting is referred to as interrupted when the tool periodically loses contact with the workpiece. Our example is taken from milling, where a rotating tool cuts a fixed workpiece. Usually, the milling tool has more than one cutting edge spaced equidistantly around its circumference. As the tool cuts, the edges periodically enter and leave the workpiece. During contact with the workpiece, the force on a cutting edge either increases or decreases monotonically, depending on the feed direction. Furthermore, depending on the number of cutting edges and the radial depth of cut, there may be no, one, or several edges simultaneously in contact with the workpiece. The forcing caused by such engagement of the cutting edges is therefore not sinusoidal. In our case, the tool had only one cutting edge (the second edge was removed for the experiment) which was in contact with the workpiece for $1/10$ of the revolution period T. This corresponds approximately to periodical impulse forcing. Details of the milling experiments can be found in Ref. [8].

Based on uni-directionally recorded displacements of the workpiece fixture, two milling regimes are inspected here: (a) chatter-free milling, a favourable regime for a machinist, and (b) chatter, an unfavourable regime. The transition between the two regimes represents a flip bifurcation [8]. [1] Fig. 3 shows the fixture motion in the reconstructed phase space during the two milling regimes. It is assumed that the underlying dynamics of the motion can be expressed by a two-dimensional equation (1) in which the noise term corresponds to the influence of the non-homogeneous material being cut. The deterministic term **h** estimated from the data was used to reconstruct the deterministic trajectories of the fixture motion shown superimposed in Fig. 3. In the chatter-free regime (Fig. 3(a)), the deterministic trajectory follows a three-loop limit cycle which indicates that the fixture–workpiece assembly completes three oscillatory cycles per tool revolution. The tool is in contact with the workpiece during the upper half of the smallest loop, located in the lower left corner of the attractor, while for the rest of the limit cycle the fixture–workpiece assembly and the tool oscillate freely. In the chatter regime (Fig. 3(b)), deterministic motion on a five-loop limit cycle is found. The tool and the workpiece come into contact twice per tool revolution, which results in the trajectory being driven from the outer to the inner loops and back once

[1] A different type of chatter, not discussed here, is reached via a Hopf bifurcation [9].

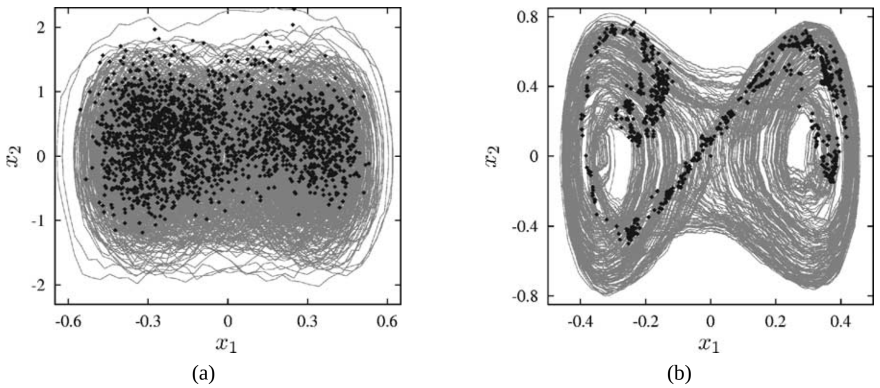

Fig. 2. (a) Original stochastic and (b) reconstructed deterministic trajectories of the Duffing oscillator in a chaotic regime with the corresponding stroboscope sections superimposed. Note the difference in scales of the figures.

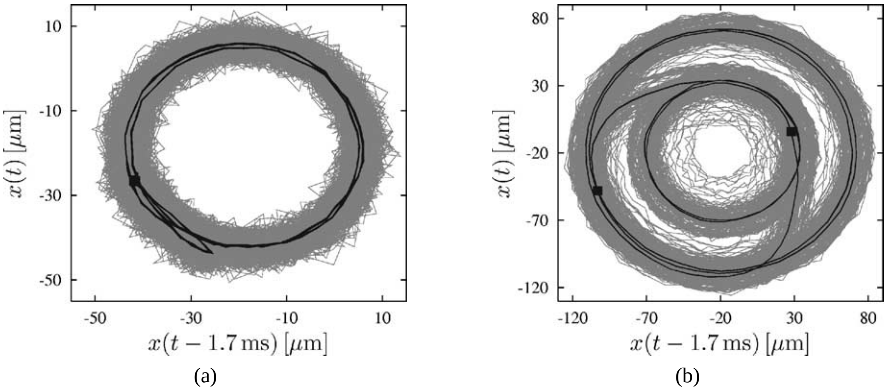

Fig. 3. Recorded trajectories of workpiece-fixture motion (grey lines) in (a) chatter-free and (b) chatter milling regimes with the corresponding reconstructed deterministic trajectories superimposed. Points of the stroboscope sections of the deterministic attractors are shown as black squares. Note the difference in scales of the figures.

per tool revolution. Also shown in Fig. 3 are the stroboscope sections of the deterministic attractors. The phase of the stroboscope with respect to the tool revolution period T was set to the presumed instant of first contact between the tool and the workpiece. As predicted by the theory of highly interrupted cutting [8,9], the chatter-free and chatter milling regimes viewed in the stroboscope section indeed correspond to period one (fixed point) and period two motion, respectively.

4. Conclusions

In this Letter, a recently proposed method for analysis of stochastic processes [2] was extended to the class of periodically forced processes. It was shown how the periodically dependent deterministic component of process dynamics can be estimated from measured data, provided that the forcing frequency is known and the data is sampled stroboscopically.

The method was illustrated using synthetic and experimental data. The first example employed data from a model describing the stochastic resonance phenomenon. Good agreement was observed between the estimates and theory. Next, data simulating large elastic deflections of a sinusoidally forced structure was analysed. The deterministic motion of the structure reconstructed from stochastic data exhibited the main features of the structure's original noise-free chaotic motion. Finally, experimental data from two different regimes of interrupted cutting (milling) was inspected. The deterministic attractors reconstructed from the data enabled a detailed study of the workpiece-fixture motion in both regimes. The results of the study agree well with the theory of interrupted cutting dynamics.

In summary, periodically forced stochastic processes are often encountered in physics, biology, engineering, economics, etc. In most cases, the forcing period is accessible and the data can be sampled stroboscopically. We believe that the method presented is an effective tool for the study of such processes.

Acknowledgement

The authors are grateful to Brian P. Mann from Washington University and Jerry E. Halley from the Boeing Company, both of St. Louis (MO, USA), for making data from interrupted cutting available.

References

[1] H. Kantz, T. Schreiber, Nonlinear Time Series Analysis, Cambridge Nonlinear Science Series, Vol. 7, Cambridge University Press, Cambridge, 1997.
[2] S. Siegert, R. Friedrich, J. Peinke, Phys. Lett. A 243 (5–6) (1998) 275.
[3] J. Gradišek, S. Siegert, R. Friedrich, I. Grabec, Phys. Rev. E 62 (3) (2000) 3146.
[4] M. Siefert, R. Friedrich, A. Kittel, J. Peinke, On a quantitative method to analyse dynamical and measurement noise (preprint).
[5] H. Risken, Fokker–Planck Equation, Springer, Berlin, 1989.
[6] L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni, Rev. Mod. Phys. 70 (1) (1998) 223.
[7] F.C. Moon, P.J. Holmes, J. Sound Vibration 65 (1979) 285.
[8] P.V. Bayly, J.E. Halley, B.P. Mann, M.A. Davies, in: Proceedings of the 18th ASME Biennial Conference on Mechanical Vibration and Noise, ASME Design and Technical Conferences, Pittsburgh, PA, 2001.
[9] T. Insperger, G. Stépán, Period. Polytech. Mech. Eng. 44 (1) (2000) 47.