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Generalized Plasticity and Uniaxial Constrained Recovery in Shape Memory Alloys

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In this article uniaxial constrained recovery is modelled using the theory of generalized plasticity, which was developed by J. Lubliner and F. Auricchio. As a mechanical obstacle that delays free recovery in a shape memory alloy wire, a bias spring made of an ordinary material is used. Two flow rules are used in the modelling: linear and exponential.

Keywords shape memory alloys, martenistic transformation, constrained recovery, generalized plasticity, linear flow rule, exponential flow rule

1. INTRODUCTION

Smart materials are receiving more and more attention, mainly for their possible innovative use in practical applications. An important example is the family of shape memory alloys (SMA), which have an intrinsic capacity to return to a previously defined shape by increasing the metal's temperature. They have an interesting property of remembering the original shape or size.

Shape memory effect (SME), arises from the changes of crystal structure of the solid phases of the material. These phases are a low-temperature phase (martensite) and a high-temperature phase (austenite). From a metallurgic point of view [1], a martensitic transformation is a solid-solid, diffusionless transition between a crystallographyically more-ordered parent phase (austenite) and a crystallographyically less-ordered product phase (martensite). In the case of SMA the martensitic transformation is reversible and, usually, rate-independent. The return to the original shape starts at a temperature called austenite start transformation temperature, A_S. This transformation completes at the austenite finish transformation temperature, A_f. Large residual strains of even 10% can be recovered in this way and the process is often refered as the free recovery. If the free recovery is hampered by an external obstacle before temperature A_f is reached, the process is called constrained recovery and large stresses, up to 800 MPa, can be generated in SMA elements. This property makes SMA ideally suited for use as fasteners, seals, connectors and clamps. Vice versa, if the SMA is cooled from fully austenitic phase, it starts to transform back to martensite at a temperature called martensite start transformation temperature, M_S. This transformation ends at the martensite finish transformation temperature, M_f. Besides the martensitic transformations associated with the thermal regime, they can be triggered by mechanical loading and the martensite obtained in this way is then called the stress-induced martensite (SIM).

In addition to the SME, the other main property of SMA is its superelastic effect (SE). At constant high temperature (above temperature A_f) a mechanical loading–unloading cycle induces highly-nonlinear large deformations. At the end of the loading-unloading cycle no permanent deformations are present. The cycle usually presents a hysteresis loop.

SMA have been studied experimentally for the last four decades and numerous constitutive models have been proposed over the last 20 years [2–13]. To the authors judgement, the theory of generalized plasticity [8, 11–13] is well suited for the modelling of complex material behaviors, SME and SE, which may occur in SMA. It is based on some fundamental axioms and on results from elementary set theory and topology.

The principal aim of the present paper is to develop a mathematical model of the uniaxial constrained recovery in SMA wire element, using the generalized plasticity theory. As a mechanical obstacle, which delays free recovery in a SMA wire, a bias spring made of an ordinary material is used. The data for Ni-Ti-6wt%Cu SMA wire and steel bias spring are fed into the mathematical model in order to generate the system response. One possible practical application of this theory is in the case of tissue fixation in minimal access surgery [14]. The process of constrained recovery is divided into three temperature regions and two flow rules [8, 11] are used in the modelling: linear and exponential. Equations for the linear flow rule are written in a closed form, but for the exponential flow rule simple numerical methods have to be used, since the solution in a closed form is not possible. In the exponential flow rule, scalar constant β is included and measures the rate at which the transformation proceeds. It is interesting to observe how by varying the constant β very different evolution processes can be obtained [12], which

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is advantage comparing to the linear model. Practical determination of constant β is not clear, so its influence on the results of constrained recovery is shown.

2. GENERALIZED PLASTICITY AND SHAPE MEMORY ALLOYS

Generalized plasticity is an internal-variable model of rate-independent inelasticity and was developed in order to model the behavior of elastic-plastic solids in which, following initial plastic loading and elastic unloading, the reloading is not necessarily elastic up to the state at which unloading began. Such solids include graphite, some stainless steels, some rocks and also SMA.

This section will discuss the application of the generalized plasticity model to a simplified representation of the behavior of the SMA. The simplification lies in the fact, which will be here ignored, that martensite, when first formed, may be present in multiple orientations (multiple-variant martensite) and is characterized by a twinned structure, which minimizes the misfit between the martensite and the surrounding austenite. On the other hand, if there is a prefered direction for the occurence of the transformation, which is often associated with a presence of stress, all the martensite crystals tend to be formed on the most favorable habit plane. The product phase is then called single-variant martensite and is characterized by a detwinned structure, which again minimizes the misfit between the martensite and surrounding austenite. During the process of constrained recovery, large stresses are generated in SMA elements and such simplification is justified.

Four functions bounding by the straight lines two bands in the stress σ-temperature T plane are introduced in the generalized plasticity model of SMA:

$$F_1 = \sigma - C(T - M_f) \tag{1}$$

$$F_2 = \sigma - C(T - M_s) \tag{2}$$

$$F_3 = \sigma - C(T - A_s) \tag{3}$$

$$F_4 = \sigma - C(T - A_f) \tag{4}$$

where C is a stress rate and is almost the same in both phase transformations (A → M and M → A). The loading surfaces are given by the lines:

$$F = \sigma - CT = konst \tag{5}$$

The geometry of the regions is shown in Figure 1.

In a two phase system, it can be assumed that the only internal variable is the fraction of mass occupied by one ofthe phases. Usually, this variable is the mass fraction of martensite ξ, with $\xi = 0$ denoting all austenite and $\xi = 1$ all martensite. The inelastic strain ε_{SR} is proportional to ξ.

where ε_{S0} is the maximum inelastic strain, attained when the solid is all martensite, and disappears when heated above A_f. Since transformation from single-variant martensite to austenite is important in the case of constrained recovery, only austenite production using two flow rules, linear and exponential, will be shown next.

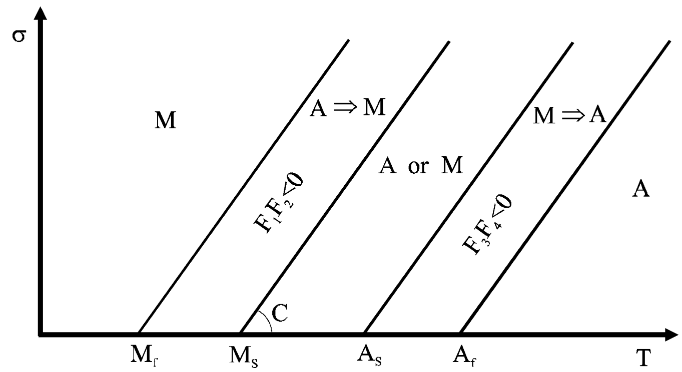

FIG. 1. Inelastic domains for austenite to martensite and martensite to austenite phase transformation.

$$\varepsilon_{SR} = \varepsilon_{S0}\xi \tag{6}$$

2.1. Austenite Production

In the region where phase transformation from martensite to austenite may take place a stress decrease at constant temperature, a temperature increase at constant stress or a proper combination of these actions should occur, Figure 2.

The conditions for phase transformation can be mathematically written in this way:

$$F_3 < 0, \quad F_4 > 0 \quad \text{and } \dot{F} < 0 \tag{7}$$

Therefore the product F_3F_4 must be negative for transformation from martensite to austenite. Now two rate equations for ξ will be examined: first the linear and then exponential flow rule type.

2.1.1. *Linear Flow Rule*

The linear flow rule for ξ can be written in the following form [11]:

$$\dot{\xi} = -\xi \frac{\langle -F_3F_4\rangle\langle -\dot{F}\rangle}{|F_3F_4|F_4} \tag{8}$$

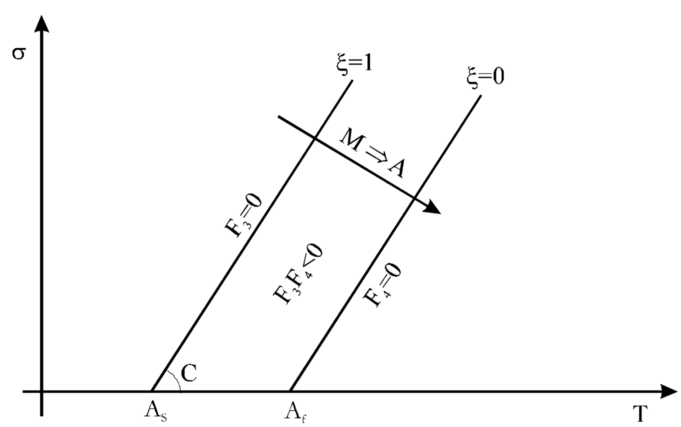

FIG. 2. Production of austenite.

where $\langle\cdot\rangle$ is the Macaulay bracket, that is $\langle x\rangle = (x + |x|)/2$. Using (3–5), expresion (8) can be rewritten in this form:

$$\frac{d\xi}{dt} = \xi\frac{\frac{d\sigma}{dt} - c\frac{dT}{dt}}{\sigma - C(T - A_{\mathrm{f}})}$$
$$\zeta = A(\sigma - C(T - A_{\mathrm{f}})) \tag{9}$$

where A is integration constant and can be defined from the condition at the beginning of the transformation, Figure 2, $\xi = 1$ and $F_3 = 0$:

$$A = \frac{1}{C(A_{\mathrm{f}}A_{\mathrm{s}})}$$
$$\xi = \frac{\sigma - C(T - A_{\mathrm{f}})}{C(A_{\mathrm{f}}A_{s})} \tag{10}$$
$$\sigma = C[T - A_{\mathrm{f}} + \xi(A_{\mathrm{f}}A_{\mathrm{s}})] \tag{11}$$

It can be seen from Eq. (10) or (11) that the relationship between martensite mass fraction ξ and stress σ is linear, so the flow rule (8) can be named linear.

2.1.2. Exponential Flow Rule

In the case of the exponential flow rule it can be written [8,11]:

$$\xi = -\beta\xi\frac{\langle -F_3F^4\rangle\langle -\dot{F}\rangle}{|F_3F_4|F_4^2} \tag{12}$$

where β is a positive constant, which measures the rate at which the phase transformation proceeds, and is another material parameter that has to be measured. Similarly as in the case of linear flow rule, it can be written:

$$\xi = B\,\exp\left[\frac{\beta}{\sigma - C(T - A_{\mathrm{f}})}\right] \tag{13}$$

where B is integration constant and can be defined from the condition at the beginning of the transformation, Figure 2, $\xi = 1$ and $F_3 = 0$:

$$B = \exp\left[\frac{\beta}{C(A_{\mathrm{f}} - A_{\mathrm{s}})}\right]$$
$$\xi = \exp\left[\frac{\beta}{C(A_{\mathrm{f}} - A_{\mathrm{s}})} - \frac{\beta}{\sigma - C(T - A_{\mathrm{f}})}\right] \tag{13}$$
$$\sigma = C(T - A_{\mathrm{f}}) + \frac{\beta C(A_{\mathrm{f}} - A_{\mathrm{s}})}{\beta - C(A_{\mathrm{f}} - A_{\mathrm{s}})\,\ln\,\xi} \tag{14}$$

It can be seen from Eq. (13) that the relationship between the martensite mass fraction ξ and stress σ is exponential, so the flow rule (12) can be named exponential.

3. PROCESS OF UNIAXIAL CONSTRAINED RECOVERY

In elements made of shape memory alloys, significant stresses occur if during heating the recovery to austenite structure is constrained by an external obstacle. The whole process is called constrained recovery. In this study the shape memory element is represented by a shape memory alloy wire (SMA wire) and the external obstacle by a linear bias spring made of conventional material. Since an SMA wire is a uniaxial element, the process is also treated as uniaxial.

The SMA wire is first cooled from austenite state, $T > A_{\mathrm{f}}$, to the martensite state, $T = T_0 < M_{\mathrm{f}}$. It is assumed that the total strain of the SMA wire in this moment is zero, $\epsilon_{\mathrm{S}} = 0$, and the length is denoted by L_0. The SMA wire is then loaded by tensile force and unloaded at constant temperature $T = T_0 < M_{\mathrm{f}}$ so that the total strain after unloading is ϵ_{S0}, and the length is denoted by L_{S0}. Strain $\varepsilon_{\mathrm{S0}} = \varepsilon_{\mathrm{SR}}$ disappears when the SMA wire is heated again above temperature A_{f}. The process is called free recovery. If recovery of the SMA wire is hampered by an external obstacle, the process is then named constrained recovery and recoverable strain $\varepsilon_{\mathrm{SR}}$ disappears only when temperature $T = T_{\mathrm{SE}} = \sigma/C + A_{\mathrm{f}}$ is reached. The process of constrained recovery is schematically shown in Figure 3, where a linear bias spring made of an ordinary material presents an external obstacle.

In Figure 3, L_{k0} is spring length at temperature T_0, T instantaneous temperature of the spring and SMA wire, L_{k} instantaneous length of the spring at temperature T, L_{S} instantaneous length of the SMA wire at temperature T, T_{C} contact temperature at which the SMA wire and the spring touch each other and α linear thermal expansion coefficient of the bias spring.

3.1. Modelling of Uniaxial Constrained Recovery

The expression for the total strain of SMA wire ε_{S} can be written in the following way:

$$\varepsilon_{\mathrm{S}} = \frac{L_{\mathrm{S}} - L_0}{L_0} \tag{15}$$

$$\begin{cases} \varepsilon_{\mathrm{S}} = \varepsilon_{\mathrm{SR}} + \varepsilon_{\mathrm{SA}} + \varepsilon_{\mathrm{SE}} \\ \quad\;\; = \varepsilon_{\mathrm{SR}} + \alpha_{\mathrm{S}}(T - T_0) + \dfrac{\sigma}{E_{\mathrm{S}}} \end{cases} \tag{16}$$

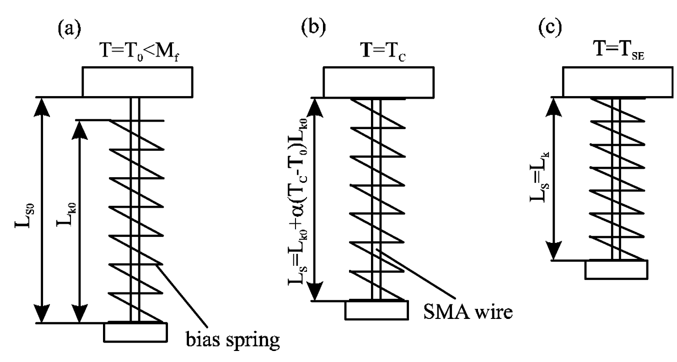

FIG. 3. Process of uniaxial constrained recovery.

where ε_{SR} is the recoverable strain of the SMA wire, ε_{SA} thermal dilatation, ε_{SE} elastic strain, α_S linear thermal expansion coefficient of the SMA wire, σ stress in the SMA wire and E_S the elastic modulus of the SMA wire. The elastic moduli E_S of austenite and martensite are usually different, but in the current approach a constant value for E_S will be chosen for both phases. The expression for the total strain of the bias spring ε_k is:

$$\varepsilon_k = \frac{L_k - L_{k0}}{L_{k0}} \tag{17}$$

The complete process of constrained recovery can be divided into three temperature ranges: a) the SMA wire has a martensite structure between the temperatures T_0 and A_S at which starts the reverse martensitic transformation in the wire. b) With increasing temperature the retransformation in the SMA wire continues (the wire contracts while the spring extends) until at temperature T_C the wire and the spring touch each other. c) Above temperature T_C, retransformation in the SMA wire is constrained because of the action of the spring and continues until temperature T_{SE} at which the retransformation in the wire is completed. The stress in the SMA wire increases, therefore the temperature T_{SE} at which the retransformation in the wire ends is equal to $\sigma/C + A_f$.

Expressions (15) and (17) hold true in all three temperature ranges.

3.1.1. *First Temperature Range* $T_0 \le T \le A_S$

The entire process of deformation of the SMA wire can be described in this way:

$$\varepsilon_S = \varepsilon_{S0} + \alpha_S (T - T_0) \tag{18}$$

With increasing temperature the spring starts extending and its total strain ε_k is:

$$\varepsilon_k = \alpha (T - T_0) \tag{19}$$

The stress states in both elements are zero, since the SMA wire and bias spring do not touch each other in this temperature ranges.

3.1.2. *Second Temperature Range* $A_S \le T \le T_C$

At temperature A_S in the SMA wire a reverse transformation into austenite is started. The deformation of the SME wire is:

$$\varepsilon_S \varepsilon_{SR} + \alpha_S (T - T_0) \tag{20}$$

Expression (20) is very similar to expression (18) only that the recoverable shape memory strain ε_{SR} of the SMA wire is no longer constant but diminishes with temperature T and can be calculated using the linear or exponential flow rule. In the case of the linear flow rule (10), assuming $\sigma = 0$ and using Eq. (6) it can be written:

$$\varepsilon_{SR} = \varepsilon_{S0} + \frac{A_f - T}{A_f - A_S} \tag{21}$$

and in the case of the exponential flow rule:

$$\varepsilon_{SR} = \varepsilon_{S0} \exp\left[-\frac{\beta (T - A_S)}{C(A_f - A_S)(A_f - T)} \right] \tag{22}$$

Recoverable contact strain of the SMA wire ε_C at temperature T_C can be calculated from the condition that the lengths of the SMA wire and the bias spring are equal, $L_S(T_C) = L_k(T_C)$:

$$\begin{cases} L_0[1 + \varepsilon_C + \alpha_S (T_C - T_0)] \\ = L_{k0}[1 + \alpha (T_C - T_0)] \end{cases} \tag{23}$$

In the case of linear flow rule (21), contact temperature T_C can now be written:

$$T_C = A_f \frac{\varepsilon_C}{\varepsilon_{S0}} (A_f - A_S) \tag{24}$$

and in the case of exponential flow rule (22):

$$T_C \frac{C(A_f - A_S) A_f \ln(\varepsilon_C/\varepsilon_{S0}) - \beta A_S}{C(A_f - A_S) \ln(\varepsilon_C/\varepsilon_{S0}) - \beta} \tag{25}$$

3.1.3. *Third Temperature Range* $T_C \le T \le T_{SE}$

In this range the deduction of expressions becomes more complicated since the stress in the SMA wire is no longer zero but increases with increasing temperature T (ε_{SR} is getting smaller and the spring resists contraction). There is a new mechanical equilibrium state which holds for the SMA wire-bias spring system, Figure 4.

The system shown in Figure 4 has to be in static equilibrium. The force in the spring F_k can be written as:

$$F_k = k[L_{k0}(1 + \alpha (T - T_0)) - L_k]; \quad F_k > 0 \tag{26}$$

where k is the spring constant. By considering the equilibrium equation, expressions (26) and (15), and equal lengths of the SMA wire and the spring, the total strain of the SMA wire ε_S can be expressed:

$$\varepsilon_S = C_1 + C_2 T - C_3 \sigma \tag{27}$$

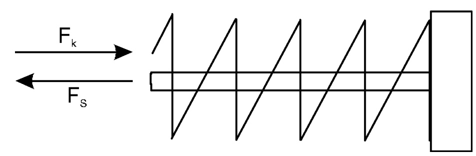

FIG. 4. Forces in the SMA wire (F_S) bias spring (F_k) system.

where C_1, C_2 and C_3 are the constants:

$$\begin{cases} C_1 = (L_{k0} - L_0 - \alpha T_0 L_{k0})/L_0 \\ C_2 = \alpha L_{k0}/L_0 \\ C_3 = Q/(kL_0) \end{cases} \tag{28}$$

and Q is the SMA wire's cross section. From Eq. (16) it can be written [15, 16]:

$$\begin{cases} d\varepsilon_S = d\varepsilon_{SR} + d\varepsilon_{SA} + d\varepsilon_{SE} \\ \qquad = d\varepsilon_{SR} + \alpha_S dT + \dfrac{d\sigma_R}{E_S} \end{cases} \tag{29}$$

where $d\sigma_R = CdT$. The stress increase $d\sigma_R$ is an elastic part of the total stress increase in the SMA wire $d\sigma_R$. It has to be emphasized that the stress increase $d\sigma_R$ is greater than the total stress increase in the SMA wire $d\sigma$, since phase transformation from martensite to austenite reduces the total stress increase $d\sigma$, Figure 5. An infinitesimal temperature increase dT induces a stress increase $d\sigma$ and a strain change $d\varepsilon_S$. This infinitesimal step from point $A_1(T, \sigma, \varepsilon_S)$ to point $A_3(T + dT, \sigma + d\sigma, \varepsilon_S + d\varepsilon_S)$, Figure 5, can be also achieved in two intermediate steps. The first step ($A_1 \to A_2$) is an infinitesimal temperature increase dT at constant recoverable strain ε_{SR}. This temperature increase yields an increase of the stress by $d\sigma_R$. The second step ($A_2 \to A_3$) is the unloading process at constant temperature $T + dT$, which results in a decrease of the total strain by a recoverable strain $d\varepsilon_{SR}$.

From (27) it can be written:

$$d\varepsilon_S = C_2 dT - C_3 d\sigma \tag{30}$$

Equations (29) and (30) can be equalized and divided by dT:

$$\frac{d\varepsilon_{SR}}{dT} = C_2 - \alpha_S - \frac{C}{E_S} - C_3\frac{d\sigma}{dT} \tag{31}$$

$$\int_{\varepsilon_C}^{\varepsilon_{SR}} d\varepsilon_{SR} = \left(C_2 - \alpha_S - \frac{C}{E_S}\right)\int_{T_C}^{T} dT - C_3\int_0^{\sigma} d\sigma$$

$$\left\{\varepsilon_{SR} = \varepsilon_C + \left(C_2 - \alpha_S - \frac{C}{E_S}\right)(T - T_C) - C_3\sigma \right. \tag{32}$$

There are two unknowns in expression Eq. (32), stress in the SMA wire σ and recoverable strain in the SMA wire ε_{SR}, therefore the linear or exponential flow rule also have to be used.

3.0.0.1. Linear flow rule. In the linear flow rule (10), relation (6) can be used:

$$\frac{d\varepsilon_{SR}}{dT} = \frac{\varepsilon_{S0}}{A_f - A_S}\left(\frac{1}{C}\frac{d\sigma}{dT} - 1\right) \tag{33}$$

$$\int_{\varepsilon_C}^{\varepsilon_{SR}} d\varepsilon_{SR} = \frac{\varepsilon_{S0}}{A_f - A_S}\left(\frac{1}{C}\int_0^{\sigma} d\sigma - \int_{T_C}^{T} dT\right)$$

$$\varepsilon_{SR} = \varepsilon_C + \frac{\varepsilon_{S0}}{C(A_f - A_S)}\sigma - \frac{T - T_C}{A_f - A_S}\varepsilon_{S0} \tag{34}$$

Equations (32) and (34) are equalized and the stress in the SMA wire σ is finally written:

$$\begin{cases} \sigma = \dfrac{C(A_f - A_S)}{\varepsilon_{S0} + CC_3(A_f - A_S)} \\ \qquad \times\left(C_2 - \alpha_3 - \dfrac{C}{E_S} + \dfrac{\varepsilon_{S0}}{A_f - A_S}\right)(T - T_C) \end{cases} \tag{35}$$

The temperature T_{SE} at which transformation from martensite to austenite during constrained recovery is completed can be derived from (32) and (34) with condition $\varepsilon_{SR}(T_{SE}) = 0$:

$$\begin{cases} T_{SE} = A_f + \frac{\varepsilon_C}{\varepsilon_{S0}} \\ \qquad \times\dfrac{\varepsilon_{S0}E_S + [E_S(C_2 - \alpha_S) - C](A_f - A_S)}{C(1 + E_S C_3) - E_S(C_2 - \alpha_S)} \end{cases} \tag{36}$$

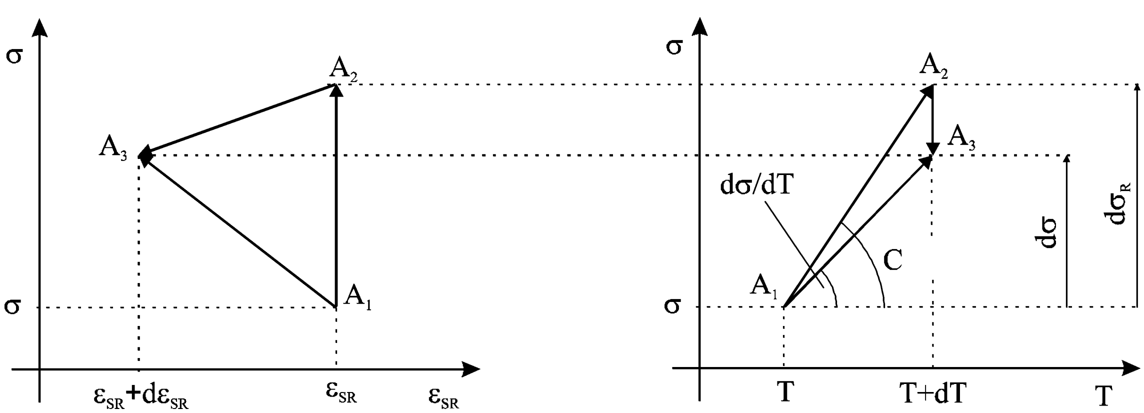

FIG. 5. The process of a temperature increase dT ($A_1 \to A_3$) can be replaced by a temperature increase dT at a constant recoverable strain ε_{SR} ($A_1 \to A_2$), followed by an unloading at constant temperature T($A_2 \to A_3$).

In the treated case the temperature T_{SE} is always greater than temperature A_f. The stress at the end of constrained recovery can also be determined using Eqs. (35) and (36) and condition $\sigma_{SE} = \sigma(T_{SE})$:

$$\begin{cases} \sigma_{SE} = \dfrac{\varepsilon_C}{\varepsilon_{S0}} C \\ \qquad \times \dfrac{\varepsilon_{S0} E_S + [E_S(C_2 - \alpha_S) - C](A_f - A_S)}{C(1 + E_S C_3) - E_S(C_2 - \alpha_S)} \end{cases} \tag{37}$$

It can clearly be seen from Eqs. (36) and (37) that $T_{SE} = A_f$ and $\sigma_{SE} = 0$, if the recoverable contact strain in the SMA wire ε_C is zero. The process is then called free recovery.

It is now possible to write the constant $C_{CR} = d\sigma/dT$ from (35) which is always positive and smaller than stress rate C:

$$\begin{aligned} C_{CR} &= \frac{d\sigma}{dT} \\ &= \frac{C(A_f - A_S)}{\varepsilon_{S0} + CC_3(A_f - A_S)} \left(C_2 - \alpha_S - \frac{C}{E_S} + \frac{\varepsilon_{S0}}{A_f - A_S} \right) \\ &= C \frac{T_{SE} - A_f}{T_{SE} - T_C} = const. \end{aligned} \tag{38}$$

It is also interesting to determine the constant $E_{CR} = d\sigma/d\varepsilon_S$, using (30) and (38), which is a kind of elasticity modulus during the process of constrained recovery:

$$E_{CR} = \frac{C_{CR}}{C_2 - C_3 C_{CR}} = const. \tag{39}$$

Constant E_{CR} depends on both elements taking part in the process of constrained recovery: the SMA wire and the spring. It can be either positive or negative depending on the properties of the SMA wire and the spring, as can be seen from (39).

Figures 6 and 7 show, in a schematic way, the entire process of constrained recovery in the SMA wire considering the linear flow rule.

From point 1 to 2, at temperature T_0, the SMA wire is subjected to tensile stress and then the stress is removed so that in point 3 the recoverable shape memory strain ε_{SR} is equal to ε_{S0}, this at the same time being the total strain of the SMA wire. From point 3 onwards, the SMA wire is heated and it extends until point 4 (temperature A_S). The process from point 3 to 4 corresponds to the first temperature range $T_0 \le T \le A_S$. In point 4 the retransformation starts in the SMA wire and goes on at a zero stress until point 5 (temperature T_C) when the wire and the spring touch each other.

The process from point 4 to 5 corresponds to the second temperature range $A_S \le T \le T_C$. From point 5 onwards the SMA wire and the spring touch each other and as a result stresses occur in both elements. With the increase in temperature, the SMA wire continues to contract, and in point 6 at temperature

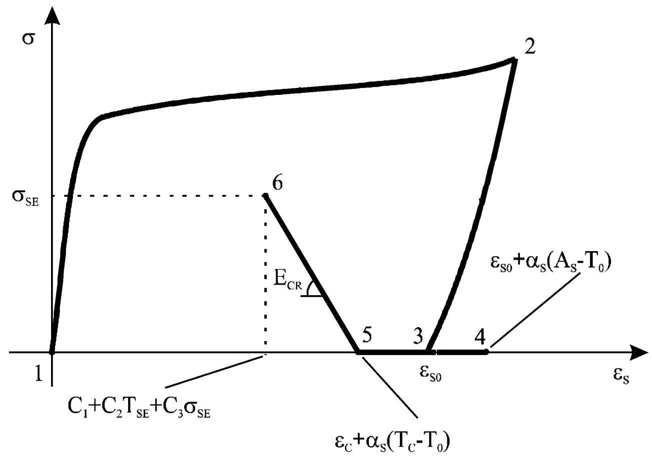

FIG. 6. Stress-strain state in the SMA wire after constrained recovery.

T_{SE} the retransformation ends. In Figure 6 the modulus of elasticity during the constrained recovery is negative: $E_{CR} < 0$. The process from point 5 to 6 corresponds to the third temperature range $T_C \le T \le T_{SE}$.

3.0.0.2. Exponential flow rule. In the exponential flow rule (14), relation (6) can be used:

$$\begin{cases} \dfrac{d\sigma}{dT} = C \dfrac{[C(A_f - A_S)]^2 \beta}{\varepsilon_{SR}[\beta - C(A_f - A_S)\ln(\varepsilon_{SR}/\varepsilon_{SO})]^2} \dfrac{d\varepsilon_{SR}}{dT} \end{cases} \tag{40}$$

Equation (40) can be equalized with expression $d\sigma/dT$ from Eq. (31) and then integrated:

$$\begin{cases} \dfrac{1}{C_3} \displaystyle\int_{\varepsilon_C}^{\varepsilon_{SR}} d\varepsilon_{SR} + [C(A_f - A_S)]^2 \beta \\ \quad + \displaystyle\int_{\varepsilon_C}^{\varepsilon_{SR}} \dfrac{d\varepsilon_{SR}}{\varepsilon_{SR}[\beta - C(A_f - A_S)\ln(\varepsilon_{SR}/\varepsilon_{S0})]^2} \\ \quad = -\left[\dfrac{C(E_S C_3 + 1)}{E_S C_3} + \dfrac{\alpha_S - C_2}{C_3} \right] \displaystyle\int_{T_C}^{T} dT \end{cases}$$

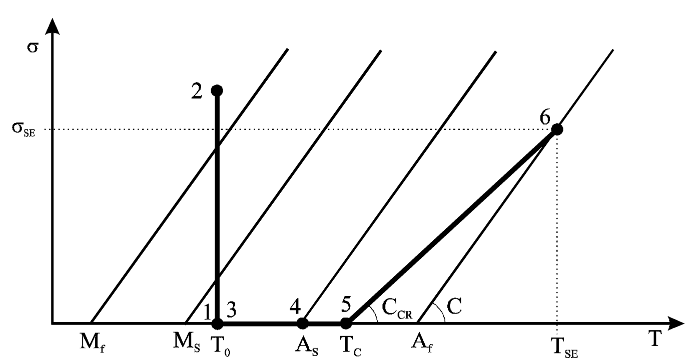

FIG. 7. Stress-temperature state in the SMA wire after constrained recovery.

$$\begin{cases} \varepsilon_{SR} - \varepsilon_C + \dfrac{\beta C_3 C(A_f - A_S)}{\beta - C(A_f - A_S)\ln(\varepsilon_{SR}/\varepsilon_{S0})} \\ -\dfrac{\beta C_3 C(A_f - A_S)}{\beta - C(A_f - A_S)\ln(\varepsilon_C/\varepsilon_{S0})} \\ +\left[C_2 - \alpha_S - \dfrac{C(E_S C_3 + 1)}{E_S}\right](T - T_C) \end{cases} \tag{41}$$

The above recoverable strain ε_{SR}-temperature T relation is not explicit; for a given temperature, iteration is needed for convergence. When recoverable strain ε_{SR} from (41) is known, the stress in the SMA wire σ at temperature T can be calculated from (32):

$$\left\{ \sigma = \frac{1}{C_3}\left[\varepsilon_C + \left(C_2 - \alpha_S - \frac{C}{E_S}\right)(T - T_C) - \varepsilon_{SR}\right] \right. \tag{42}$$

Even though the recoverable strain ε_{SR} and the stress in the SMA wire at a temperature T cannot be written in a closed form, the temperatute T_{SE} and the stress σ_{SE} can be written in a closed form. If $\varepsilon_{SR}\ (T_{SE}) = 0$ is set into Eq. (41), the temperature T_{SE} at which constrained recovery is completed can be determined:

$$\begin{cases} T_{SE} = T_C + \dfrac{E_S}{C(E_S C_3 + 1) - E_S(C_2 - \alpha_S)} \\ \quad \times \left[\dfrac{\beta C_3 C(A_f - A_S)}{\beta - C(A_f - A_S)\ln(\varepsilon_C/\varepsilon_{S0})} + \varepsilon_C\right] \end{cases} \tag{43}$$

Using (43) in (42) and according to condition $\sigma_{SE} = \sigma(T_{SE})$, the stress at the end of the constrained recovery σ_{SE} can also be determined:

$$\begin{cases} \sigma_{SE} = \dfrac{C - E_S(C_2 - \alpha_S)}{E_S(C_2 - \alpha_S) - C(E_S C_3 + 1)} \\ \quad \times \left(\dfrac{\beta C(A_f - A_S)}{\beta - C(A_f - A_S)\ln(\varepsilon_C/\varepsilon_{S0})} + \dfrac{\varepsilon_c}{C_3}\right)\dfrac{\varepsilon_C}{C_3} \end{cases} \tag{44}$$

It is clearly seen that the process of stress generation and strain recovery is quite complex, even in the case of uniaxial elements. In the case of the exponential flow rule the solution cannot be written in a closed form. Moreover, this complex behaviour is

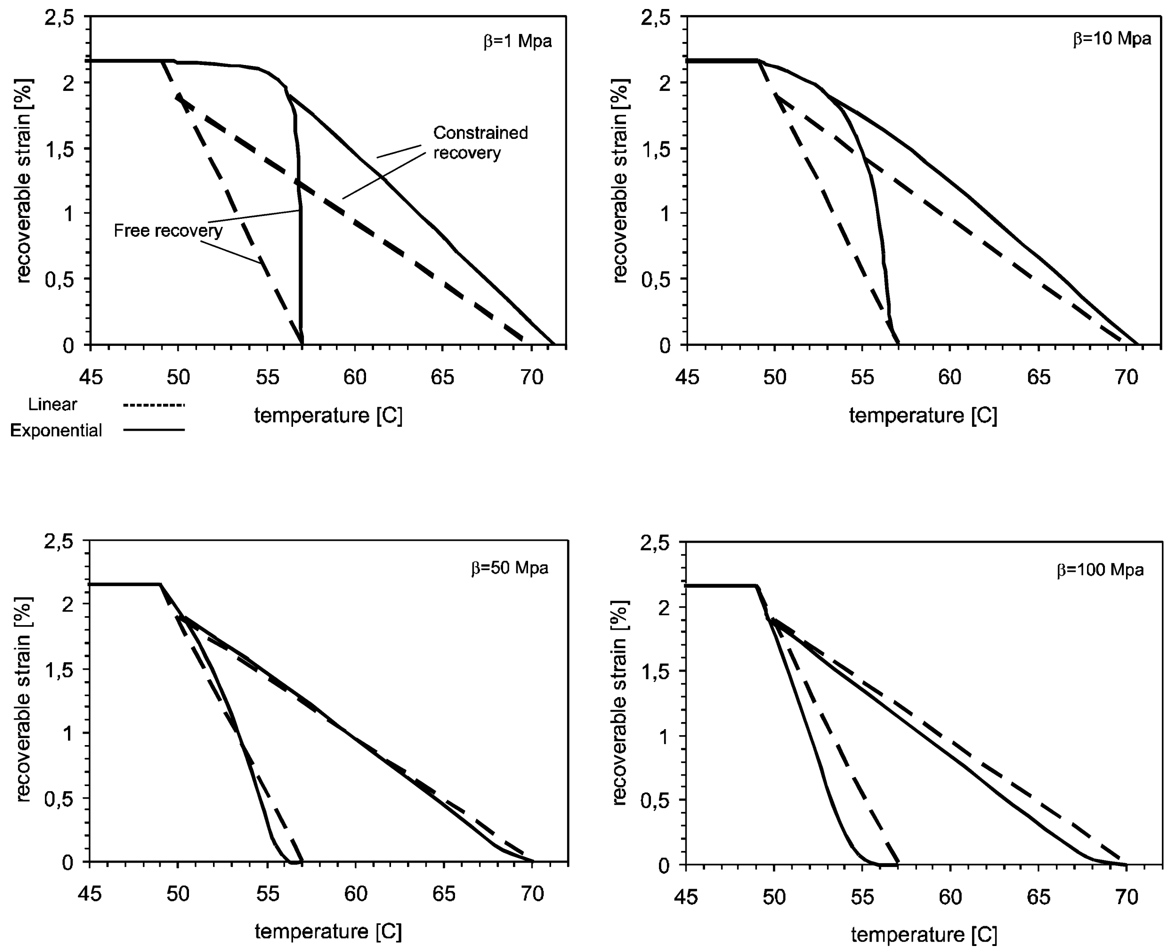

FIG. 8. The relationship between temperature T and recoverable strain in the SMA wire ε_{SR} at four different values of the constant β =1, 10, 50, 100 MPa and contact strain ε_C =1.9 %.

influenced by a large number of parameters and practical determination of one of these parameters, constant β, is not clearly known.

4. NUMERICAL EXAMPLES

The numerical values for material parameters are based on those reported in literature [16, 17] for a Ni-Ti-6wt%Cu alloy SMA wire with a diameter of 1.15 mm and a steel bias spring (though not exactly equal to):

$A_S = 49°C$	$A_f = 57°C$	$T_0 = 30°C$
$E_S = 27000$ MPa	$\varepsilon_{S0} = 2.155\%$	$\varepsilon_C = 1.9\%$
$Q = 1.0387$ mm^2	$L_{S0} = 100$ mm	$C = 9.83$MPa/K
$\alpha_S = 6.6 \times 10^6 k^{-1}$	$\alpha = 1 \times 10^{-1} k^{-1}$	$k = 100$ N/mm

Figure 8 shows the relationship between temperature T and recoverable strain in the SMA wire ε_{SR} at four different values of constant β. As mentioned before, the way of a determination of the constant β is not known yet. The results of the free and constrained recovery using both flow rules, linear and exponential, are shown and are quite similar for $\beta = 50$ MPa, while for $\beta = 1$ and 10 MPa are not. Despite this resemblance it can not be concluded for sure, that for this particular SMA alloy the constant β may be approximately 50 MPa.

Figure 9 shows the relationship between the temperature T and the stress in the SMA wire σ at the same values of the constant β as in Figure 8 during the process of constrained recovery. Again, the results for the linear and exponential flow rules are quite similar for $\beta = 50$ MPa.

It is interesting to note the decrease in stress at the end of constrained recovery for $\beta = 50$ and 100 MPa. This happens because the phase transformation from martensite to austenite slows down for these two values of β at the end of the process of constrained recovery, Figure 8, but SMA wire is expanding at the same time due to the linear thermal expansion process, which is equal in all three temperature domains. The combination of both processes causes some stress relaxation at the end of the process in the SMA wire. This effect is not seen if the linear flow rule or exponential flow rules with $\beta = 1$ and 10 MPa are used. The contact temperature T_C in Figures 8 and 9 for the linear flow rule is 49.95°C, while for exponential flow rule this temperature varies with positive constant β. It is also clearly seen that the temperature T_{SE} at which the process of constrained recovery ends is much higher than the temperature A_f at which phase

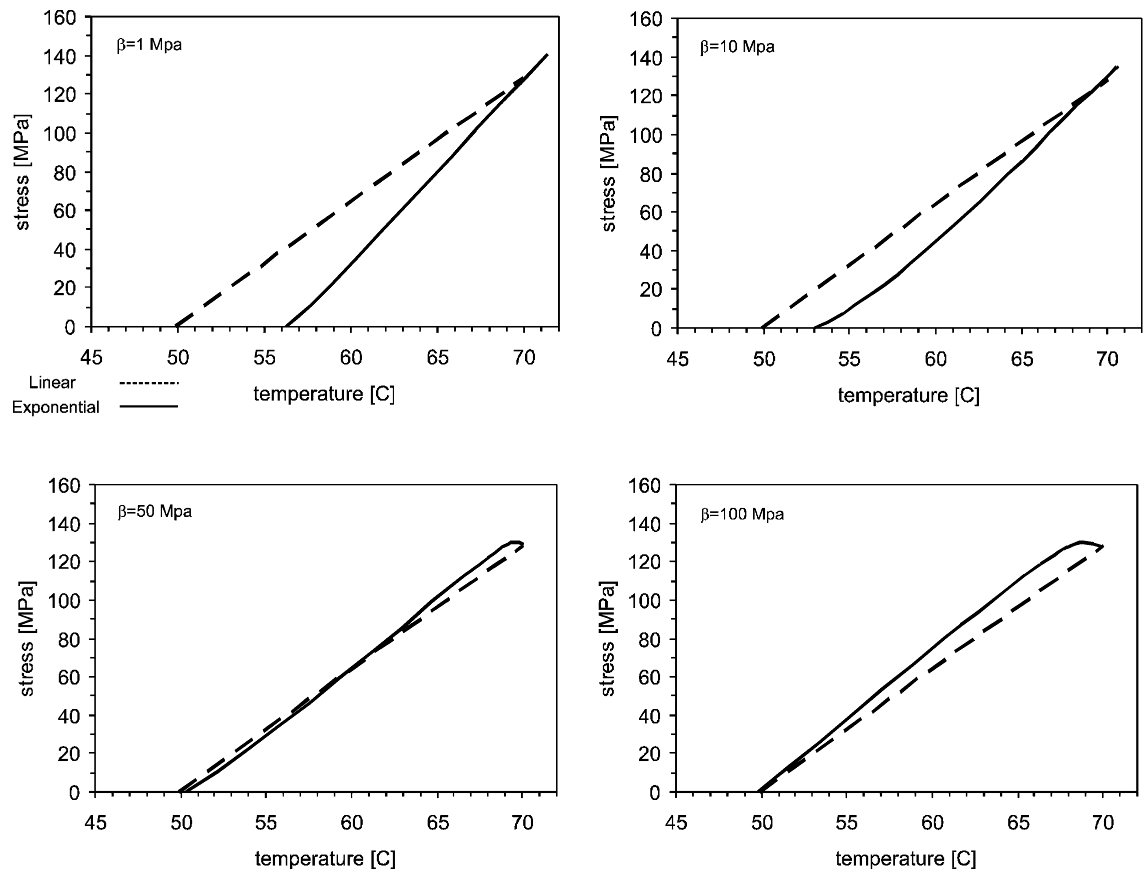

FIG. 9. The relationship between the temperature T and the stress in the SMA wire σ at four different values of the constant $\beta = 1, 10, 50, 100$ MPa and contact strain $\varepsilon_C = 1.9\%$.

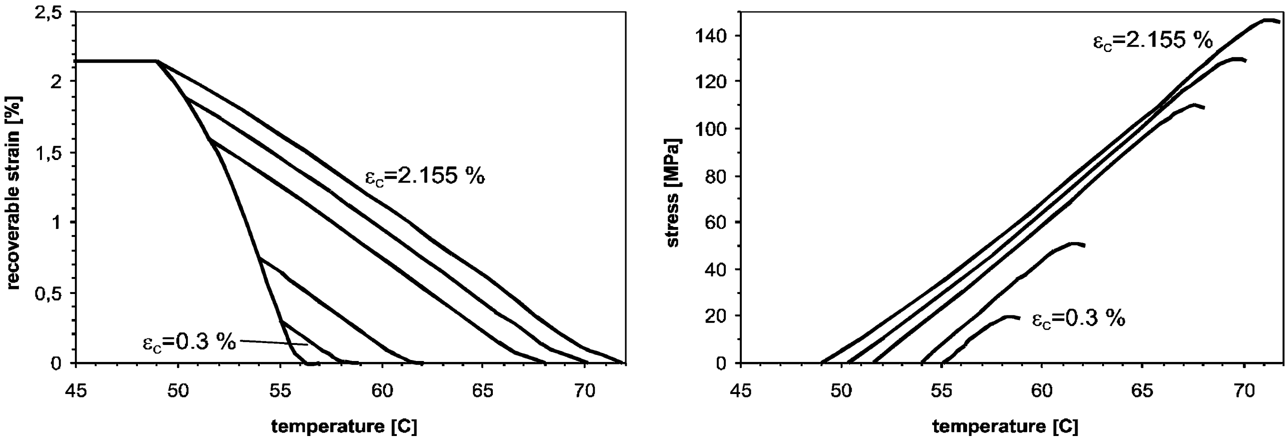

FIG. 10. Recoverable strain in the SMA wire ε_{SR} versus temperature T and stress in the SMA wire σ versus temperature T using $\beta = 50$ MPa at five different contact strains: $\varepsilon_C = 0.3\%, 0.75\%, 1.6\%, 1.9\%$ and 2.155%.

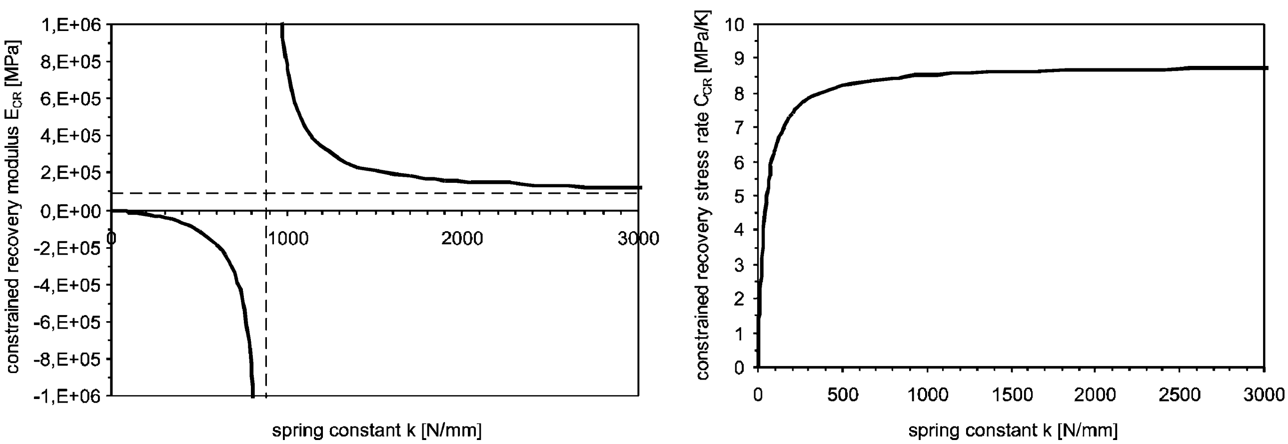

FIG. 11. Spring constant k versus stress σ_{SE} and spring constant k versus temperature T_{SE} using exponential flow rule $\beta = 50$ MPa and contact strain $\varepsilon_C = 1.9\%$.

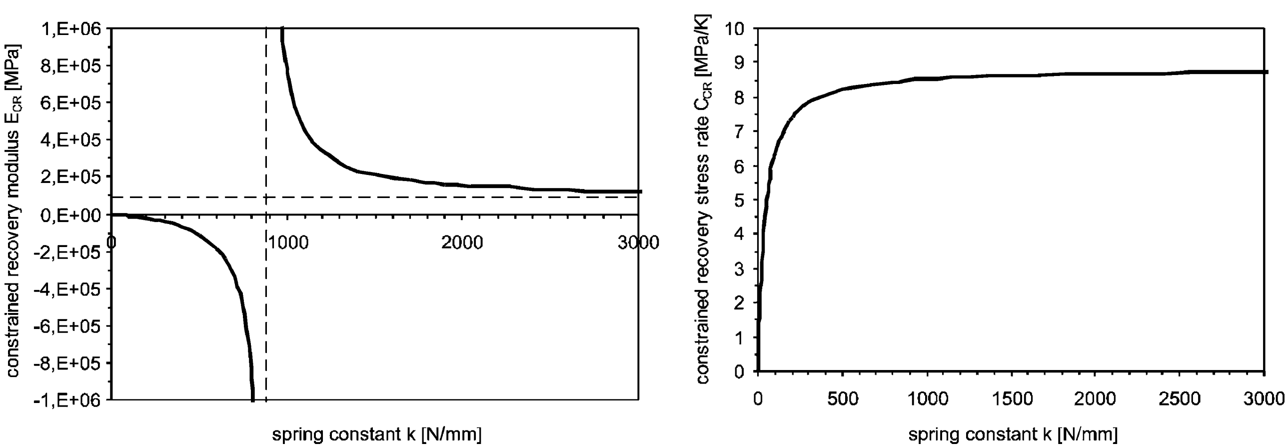

FIG. 12. Spring constant k versus constrained recovery modulus E_{CR} and spring constant k versus constrained recovery stress rate C_{CR} using linear flow rule and contact strain $\varepsilon_C = 1.9\%$.

transformation from martensite to austenite ends at zero stress. The slope $C_{CR} = d\sigma/dT$ is similar in all four cases for different constants β and is a function of temperature T, while for linear flow rule it is a constant, as can be seen in Figure 9.

Figure 10 presents the relationship between recoverable strain in the SMA wire ε_{SR} and temperature T and between stress in the SMA wire σ and temperature T at five different contact strains in the SMA wire: $\varepsilon_C = 0.3\%, 0.75\%, 1.6\%, 1.9\%, 2.155\%$. The exponential flow rule with $\beta = 50$ MPa is used in the calculation. It is clearly seen that stresses are highest, if the contact strain $\varepsilon_C = \varepsilon_{S0} = 2.155\%$ and the SMA wire and the bias spring touch each other at temperature $T_C = A_S = 49°C$, when phase transformation in the SMA wire begins. Again, the decrease of the stress at the end of constrained recovery can be noted and the reason for the phenomenon is the same as before.

Figure 11 shows the relationship between the spring constant k and the stress σ_{SE} at the end of the process of constrained recovery in the SMA wire and between the spring constant k and the temperature T_{SE} at which constrained recovery ends. The exponential flow rule with $\beta = 50$ MPa is used in the calculation. It can be clearly seen that the stress σ_{SE} and the temperature T_{SE} are limiting to a specific value as spring constant k is increasing. If spring constant $k = \infty$, then $\sigma_{SE} = 628.77$ MPa, $T_{SE} = 120.96°C$.It is interesting to note what extreme stresses can be generated by the SMA wire, even though its elastic modulus E_S is relatively low. This is one of the many unique properties of the SMAs.

Figure 12 shows the relationship between spring constant $k$versus constrained recovery modulus $E_{CR} = d\sigma/d\varepsilon_S$ defined in (39) and spring constant k versus constrained recovery stress rate $C_{CR} = d\sigma/dT$ defined in (38). The linear flow rule is used in both graphs.

As already mentioned, constant E_{CR} can be positive or negative, and for $k = 884.38$ N/mm it is $E_{CR} = \infty$ ($L_S =$ const. during the process of constrained recovery). Constant C_{CR} is also limiting to an exact value: $C_{CR} = 8.85$ MPa/K at $k = \infty$ and is always smaller than stress rate $C = 9.83$ MPa/K.

5. CONCLUSIONS

In this paper the process of constrained recovery in a SMA wire is dealt with using the theory of generalized plasticity. Even though simple flow rules were chosen, the problem can be solved in a closed form only in the case of linear flow rule, while in the case of exponential flow rule the solution in closed form is not possible. This fact clearly shows that mathematical modelling of smart structures is usually more complicated than in the case of ordinary materials. A comprehensive study concerning determination of the constant β should be carried out in the future. The results obtained with the linear flow rule are the same as the results obtained in a different way [17] by the present authors.

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