Non-prismatic Non-linearly Elastic Cantilever Beams Subjected to an End Moment

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ABSTRACT: This paper analyzes large deflection profiles of slender, inextensible cantilever beams of prismatic and non-prismatic longitudinal shapes with rectangular cross-sections subjected to a concentrated moment at the free end. The stress–strain relationship of the material is represented by the Ludwick constitutive law. Different non-linear stress–strain relations in tensile and compressive domain are considered. The main purpose of this paper is to investigate the influence of geometrical and material non-linearities on the shape of the deflection curve. The solution of a strongly non-linear set of equations is obtained numerically. In the case when non-linear stress–strain relationship in tensile and compressive domain is identical, an analytical solution is given in terms of infinite series. Several examples for a variety of different shapes of beams are presented considering both linearly and non-linearly elastic materials in the elastic domain.

KEY WORDS: large deflections, non-prismatic cantilever beam, end moment, Ludwick formula, material non-linearity.

INTRODUCTION

THE SUBJECT OF large deflection of flexible beams and elastica problems still attracts considerable attention from many scientists. The mathematical model of elastica is based on assumptions that the beam is inextensible, shear stresses are negligible in comparison with the normal stresses when the length-to-height ratio of the beam is large, and Bernoulli hypothesis which states that plane cross-sections, which are perpendicular to the neutral axis before deformation, remain plain and perpendicular to the neutral axis after deformation and do not change their shape and area. It then follows that the Euler–Bernoulli equation, which states that local bending moment is solely proportional to the local curvature, is valid.

Most of the studies in elastica theory have focused on problems of linearly elastic material. There are a lot of papers, and without even attempting to give an exhaustive list, a few examples can be found in references [1–5].

However, studies of large deflections of slender beams made of non-linearly elastic material are less frequent. Literature that is most relevant to the problem discussed here follows.

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Oden and Childs [6] studied the post-buckling problem of finite deflections of a clamped-free column subjected to an axial force and constructed of a non-linearly elastic material characterized by a moment-curvature law similar to that exhibited by a class of elastoplastic materials. Prathap and Varadan [7] investigated the inelastic large deformation of a uniform cantilever of rectangular cross-section with a vertical tip load at the free end. The material is assumed to have a stress–strain relationship of the Ramberg–Osgood type. Lo and Das Gupta [8] examined bending of a non-linear rectangular beam in large deflections. For the sections of the beam in the elastic domain, the linear stress–strain relationship is used, and in the domain where maximum stress exceeds the elastic limit, the stress is represented by a logarithmic function of strain. Lewis and Monasa [9] solved the problem of large deflections of a prismatic cantilever beam made of Ludwick type materials subjected to an end moment. The results are given for the vertical and horizontal deflections at the free end of the beam for rotations less than $\pi/2$. Fertis and Lee [10] developed a thoughtful analysis of flexible non-prismatic bars subjected to complicated loading conditions using simplified non-linear equivalent systems. The research on this matter is treated in more detail in reference [11]. Lee [12] considered the large deflection problem of a prismatic cantilever beam made of Ludwick type material under a combined loading consisting of a uniformly distributed load and vertical concentrated load at the free end. Baykara et al. [13] studied the effect of bimodulus material behavior on the horizontal and vertical deflections of a thin cantilever beam under an end moment. Jung and Kang [14] analyzed buckling of a prismatic, inextensible column fiber which is considered to be with no shear effect and whose constitutive equation corresponds to a Ludwick or modified Ludwick type. They presented solutions for four different combinations of a horizontal and verical direction of point and distributed load. Anandjiwala and Gonsalves [15] investigated the effects of fabric bending parameters on the buckling behavior of woven fabrics by numerical computations.

The main purpose of the present paper is to investigate the influence of geometrical (large deflections, non-prismatic shape of the beam) and material non-linearities (non-linear stress–strain relation, different material constants in tension and compression) on the shape of the deflection curve.

FORMULATION OF THE PROBLEM

The mathematical model of the discussed problem is based on elastica theory. We consider a slender, initially straight cantilever beam of length L subjected to a moment M_e which is applied at the free end of the beam, as shown in Figure 1. The Cartesian (x, y)-coordinate system is chosen in such a manner that the abscissa axis coincides with the axis of the undeformed beam and the coordinate origin is located at its clamped end.

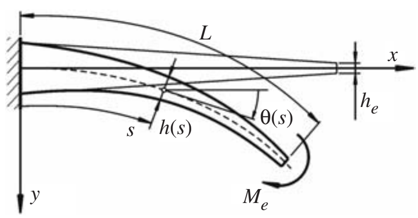

Figure 1. *Bending of non-prismatic cantilever beam.*

Parameter s, $0 \leq s \leq L$, denotes the curvilinear coordinate along the axial line measured from the clamped end and $\theta(s)$ the angle between the positive x-axis and the tangent to the neutral axis at point s. The cantilever beam has a rectangular cross-section, and a variable height, where $h(s)$ denotes height of cross-section at point s, a constant width, and h_e the height of the beam at the free end. The material of which the beam is made is of non-linearly elastic material for which the experimental stress–strain relation is represented by the Ludwick constitutive law, i.e.:

$$\sigma = \text{sign}(\varepsilon) E |\varepsilon|^{1/k} \tag{1}$$

where σ and ε are the stress and strain, respectively, E and k are material constants, Figure 2.

The function sign in this particular case indicates that Ludwick formula is valid in tension and compression. Additionally, different non-linear stress–strain relations in the tensile and compressive domain are considered.

BASIC EQUATIONS

The concepts and assumptions stated in the previous sections will serve as a starting point for derivation of governing equations. An infinitesimal element of the deflected beam is shown in Figure 3.

The equilibrium of bending moments yields:

$$\text{d}M = 0 \tag{2}$$

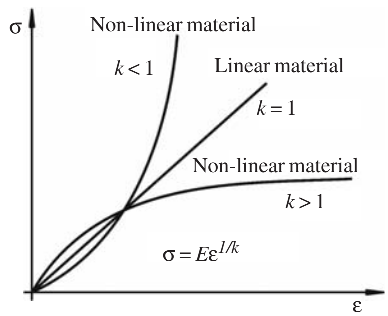

Figure 2. *Stress–strain relation for Ludwick type material in tensile domain.*

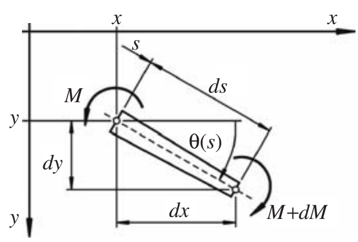

Figure 3. *Infinitesimal element of the deflected beam.*

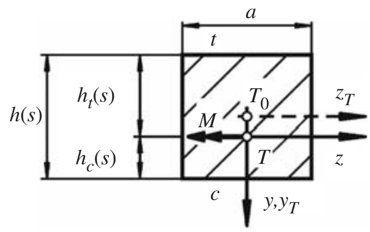

Figure 4. *Rectangular cross-section of the beam.*

The boundary conditions for the cantilever beam in Figure 1 are:

$$\theta(s = 0) = 0 \tag{3}$$

$$M(s = L) = M_e. \tag{4}$$

Additionally:

$$y(s = 0) = 0 \tag{5}$$

$$x(s = 0) = 0. \tag{6}$$

By integrating Equation (2) and using the boundary condition (4), we get

$$M(s) = M_e. \tag{7}$$

Next, Figure 4 shows a rectangular cross-section of the beam, where T_0 and T are points on the centroidal and neutral axis, respectively.

Since different material constants are applied in tension and compression, the neutral and the centroidal axis no longer coincide. As shown in Figure 4, a neutral line is a line at position $h_t(s)$ from the maximum in the tension surface, and $h_c(s)$ from the maximum in the compression surface of the beam. It can be noted that

$$h_t(s) + h_c(s) = h(s). \tag{8}$$

An undeformed longitudinal shape of the beam is defined by the equation:

$$h(s) = h_e\left(\frac{1-p}{L}s + p\right)^q \tag{9}$$

where p and q are the coefficients that determine the shape of the cantilever.

It is known that the inner bending moment, acting at any cross-section of the beam, can be expressed with normal stress σ, as:

$$M = -\int_A \sigma y \mathrm{d}A \tag{10}$$

where σ is related to corresponding strain in tension and compression, i.e.:

$$\sigma_t = E_t|\varepsilon_t|^{1/k_t} \tag{11}$$

$$\sigma_c = -E_c|\varepsilon_c|^{1/k_c} \tag{12}$$

where t and c denote the tensile and compressive domains, respectively, and E_t, E_c, k_t, and k_c material constants in corresponding domains. Combining Equations (7)–(12), taking into account that radius of curvature $\rho(s) \geq 0$ for all s, where $0 \leq s \leq L$, and using the normal strain-curvature expression:

$$\varepsilon = \rho^{-1} y \tag{13}$$

Equation (10) takes the form

$$M_e = \frac{E_t I_t(s)}{\rho(s)^{1/k_t}} + \frac{E_c I_c(s)}{\rho(s)^{1/k_c}} \tag{14}$$

where:

$$I_t(s) = \frac{k_t}{2k_t + 1} h_t(s)^{(2k_t+1)/k_t} a \tag{15}$$

$$I_c(s) = \frac{k_c}{2k_c + 1} h_c(s)^{(2k_c+1)/k_c} a. \tag{16}$$

In Equations (8) and (14) three unknowns can be identified, i.e., $h_t(s)$, $h_c(s)$ and $\rho(s)$. To obtain a complete solution an additional equation is needed. Given that there are no axial loads applied to the beam, the total axial force generated by the normal stresses must be zero. This can be expressed as:

$$\int_A \sigma \, \mathrm{d}A = 0 \tag{17}$$

Substituting Equations (8) and (11)–(13) into Equation (17), results in:

$$\frac{E_t S_t(s)}{\rho(s)^{1/k_t}} + \frac{E_c S_c(s)}{\rho(s)^{1/k_c}} = 0 \tag{18}$$

where:

$$S_t(s) = \frac{k_t}{k_t + 1} h_t(s)^{(k_t+1)/k_t} a \tag{19}$$

$$S_c(s) = -\frac{k_c}{k_c + 1} h_c(s)^{(k_c+1)/k_c} a \tag{20}$$

Variables $h_t(s)$, $h_c(s)$, and $\rho(s)$ are calculated at discrete locations on the deflected beam when solving the system of Equations (8), (14), and (18). The solution was obtained using Newton–Raphson iterative method. Finally, by using the expression for curvature:

$$\frac{1}{\rho(s)} = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} \tag{21}$$

and boundary conditions (3)–(6), geometric relations from Figure 3, i.e.:

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta(s) \tag{22}$$

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta(s) \tag{23}$$

can be employed to obtain the Cartesian coordinates for a point on the neutral axis of the deflected non-linear cantilever beam.

In the case when $E_t = E_c = E$ and $k_t = k_c = k$ it is possible to obtain an analytical solution to the problem of non-linear large deflections of non-prismatic cantilever beam in terms of infinite series. Equation (14) thus becomes:

$$M_e = \frac{EI(s)}{\rho(s)^{1/k}} \tag{24}$$

where:

$$I(s) = \left(\frac{1}{2}\right)^{(k+1)/k} \frac{k}{2k+1} h(s)^{(2k+1)/k} a \tag{25}$$

Using Equations (9), (21), (24), and (25), we can find:

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = u(rs+p)^w \tag{26}$$

where:

$$u = \frac{M_e^k}{E^k \left(\frac{1}{2}\right)^{k+1} \left(\frac{k}{2k+1}\right)^k h_e^{2k+1} a^k}, \quad r = \frac{1-p}{L}, \quad w = -q(2k+1). \tag{27}$$

Integrating Equation (26) and applying boundary condition (3), leads to:

$$\begin{aligned} \theta(s) &= u \int_0^s (rt+p)^w \, \mathrm{d}t \\ &= m(rs+p)^{w+1} - mp^{w+1} \end{aligned} \tag{28}$$

where:

$$m = \frac{u}{r(w+1)}. \tag{29}$$

Substituting Equation (28) into Equation (22), taking into account a trigonometric addition formula for the sine function, and expanding the derived expression into an infinite series results in:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}s} &= \cos\left(mp^{w+1}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(m(rs+p)^{w+1}\right)^{2n+1}\\ &\quad - \sin\left(mp^{w+1}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(m(rs+p)^{w+1}\right)^{2n}.\end{aligned} \tag{30}$$

Further, integrating Equation (30) and considering boundary condition (5) yields:

$$\begin{aligned}y(s) &= \cos(mp^{w+1})\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,\frac{m^{2n+1}\left((rs+p)^{(w+1)(2n+1)+1} - p^{(w+1)(2n+1)+1}\right)}{((w+1)(2n+1)+1)r}\\ &\quad - \sin(mp^{w+1})\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,\frac{m^{2n}\left((rs+p)^{2n(w+1)+1} - p^{2n(w+1)+1}\right)}{(2n(w+1)+1)r}.\end{aligned} \tag{31}$$

In a similar way, $x(s)$ is deduced, i.e.:

$$\begin{aligned}x(s) &= \cos(mp^{w+1})\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,\frac{m^{2n}\left((rs+p)^{2n(w+1)+1} - p^{2n(w+1)+1}\right)}{(2n(w+1)+1)r} +\\ &\quad + \sin(mp^{w+1})\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,\frac{m^{2n+1}\left((rs+p)^{(w+1)(2n+1)+1} - p^{(w+1)(2n+1)+1}\right)}{((w+1)(2n+1)+1)r}\end{aligned} \tag{32}$$

RESULTS AND DISCUSSION

To illustrate the influence of the parameters k_t and k_c, several examples with different configurations of material constants are discussed, cf. Table 1. In nature, if a material has non-linear response in tension it is likely that the response in compression will also be non-linear (k_t, $k_c \neq 1.0$). Therefore, it should be noted that combinations t_1–t_5 in which at

Table 1. Material constants of Ludwick type non-linear material.

Configuration	t_1	t_2	t_3	t_4	t_5	t_6	t_7
k_t	1.0	1.8	1.0	0.8	1.0	1.8	0.8
k_c	1.0	1.0	1.8	1.0	0.8	0.8	1.8
	$E_t = 125 \times 10^3$ MPa		$E_c = 75 \times 10^3$ MPa				

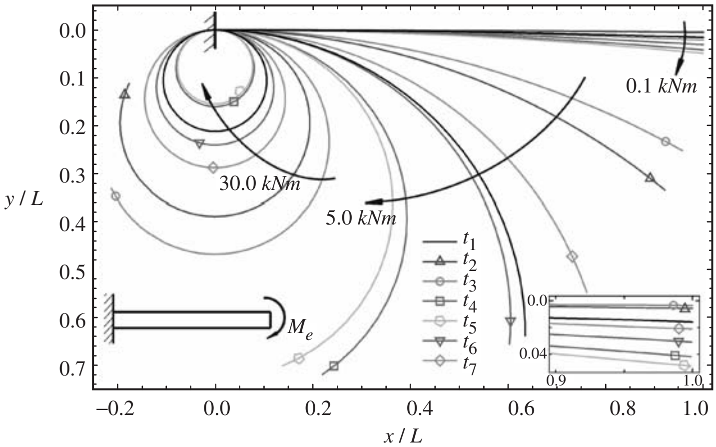

Figure 5. *Deflections of the beams for Example 1.*

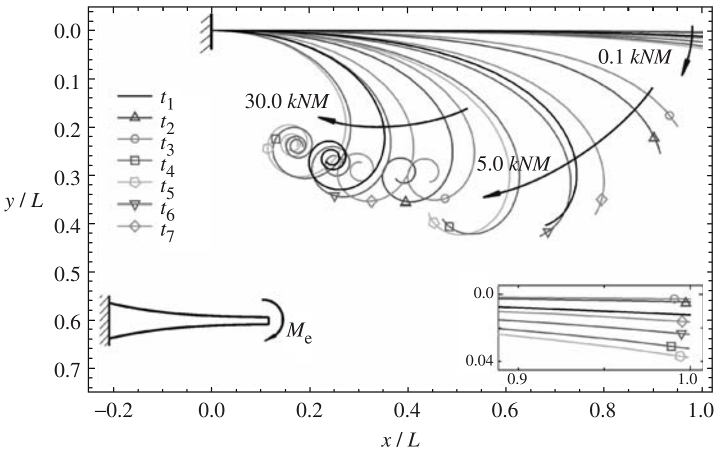

Figure 6. *Deflections of the beams for Example 2.*

least one of tensile or compressive domains (of the beams) consists of linearly elastic material are selected only to better understand the influence of non-linearity parameters on behavior of the beam. Numerical and analytical results for the treated cases are presented in Figures 5–7 and Tables 3–6.

Geometrical properties of the cantilever beam and the coefficients that determine the longitudinal shape of the cantilever beam are given in Table 2.

In the following numerical examples, cantilever beams are subjected to several different end moments. Since it is obvious that an increasing modulus of elasticity in tension or compression decreases deformations, no variation of moduli E_t, E_c is made.

Example 1

As the first example, a case of prismatic cantilever beam, cf. Table 2, of linearly elastic and Ludwick type non-linearly elastic material is shown in Figure 5.

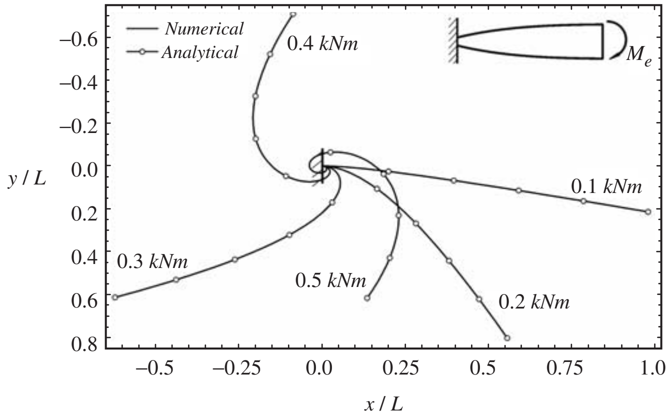

Figure 7. *Deflections of the beam for Example 3.*

Table 2. Geometrical properties and coefficients that determine the shape of the beam, cf. Equation (9).

Example	*a* (m)	h_e (m)	*L* (m)	*p*	*q*
1	0.05	0.02	1.0	0.5	0.0
2	0.05	0.01	1.0	1.4	4.0
3	0.05	0.03	1.0	0.1	0.4

Table 3. Comparison of end deflections for prismatic beam made of annealed copper.

	Lewis and Monasa [9]		Present study (analytical)	
M_e (Nm)	δ_h (mm)	δ_v (mm)	δ_h (mm)	δ_v (mm)
1.1298	0.010	2.807	0.0104	2.8098
2.2597	0.206	12.545	0.2069	12.5535
3.3895	1.191	30.089	1.1915	30.1071
4.5194	4.117	55.850	4.1213	55.8802
5.6492	10.752	89.842	10.7630	89.8873
6.7791	23.454	131.506	23.4768	131.5678
7.9089	45.044	179.418	45.0861	179.4925
9.0388	78.504	230.932	78.5701	231.0162
10.1686	126.418	281.889	126.5154	281.9756

Table 4. Convergence of series (31) and (32) for a prismatic annealed copper beam, for $M_e = 6.7791$ Nm.

No. of terms	1	2	3	4	5
$x(s = L)$	33.1116	23.2161	23.4808	23.4767	23.4768
$y(s = L)$	135.0994	131.4584	131.5697	131.5678	131.5678

Table 5. Comparison of deflections for non-prismatic annealed copper beam in example 3.

	Present study (numerical)			Present study (analytical)		
M_e (kNm)	0.1	0.3	0.5	0.1	0.3	0.5
$x(s=L/3)$	0.3286	−0.0500	0.1436	0.3286	−0.0501	0.1435
$y(s=L/3)$	0.0529	0.2761	−0.0159	0.0529	0.2762	−0.0158
$x(s=2L/3)$	0.6528	−0.3200	0.2262	0.6528	−0.3202	0.2258
$y(s=2L/3)$	0.1305	0.4696	0.2966	0.1305	0.4695	0.2968
$x(s=L)$	0.9756	−0.6210	0.1348	0.9756	−0.6212	0.1340
$y(s=L)$	0.2134	0.6125	0.6156	0.2134	0.6123	0.6157

Table 6. Convergence of series (31) and (32) for non-prismatic annealed copper beam in example 3 for $M_e = 0.5$ kNm.

No. of terms	5	9	13	15
$x(s=L/3)$ (m)	2.16796	0.157293	0.143609	0.143605
$y(s=L/3)$ (m)	−3.63448	−0.031988	−0.015802	−0.0157989
$x(s=2L/3)$ (m)	2.250120	0.239454	0.225770	0.225766
$y(s=2L/3)$ (m)	−3.32186	0.280638	0.296823	0.296827
$x(s=L)$ (m)	2.158390	0.147724	0.13404	0.134036
$y(s=L)$ (m)	−3.00293	0.599559	0.615744	0.615748

The effect of material non-linearity is evident from the difference in deflection curves in Figure 5. It can be seen that for combinations t_1–t_5 increasing the vlaue of material constant k_t or k_c results in decreasing the deformation and vice versa. Also, from comparing the deflection curves for combinations t_2 and t_3 (or t_4 and t_5) we can see that moduli of elasticity E_t and E_c also have an important influence. The deformation of the t_3 beam is considerably smaller than of the t_2 beam because the value of modulus E_t is greater than E_c, see Table 1 and Equations (11) and (12). The same is valid for t_4 and t_5, which is also in agreement with conclusions of Baykara et al. [13]. For comparison the deflection of the linear beam is also presented in Figure 5. Interesting results are obtained when neither k_t nor k_c is equal to 1.0, i.e., neither tensile nor compressive section of the beam is made of linearly elastic material. As previously observed, the deformation of t_7 is smaller than of t_6 because the value of modulus E_t is greater than E_c. Moreover, it follows from Figure 5 that the deformation of the beam from linearly elastic material t_1 is smaller than t_6 and t_7 when subjected to $M_e = 0.1$ kNm. When it is subjected to $M_e = 5.0$ kNm the deformation of t_1 is between t_6 and t_7, and when subjected to $M_e = 30.0$ kNm the deformation of t_1 is greater than t_6 and t_7.

The analytical solution, Equations (31) and (32), was verified for the case of prismatic cantilever beam made of Ludwick type annealed copper material, represented by parameters $E = 458.501$ MPa, $k = 0.463$. The results of horizontal and vertical end deflections, $\delta_h = L - x(s = L)$ and $\delta_v = y(s = L)$, are given in Table 3.

It is clearly seen that the results are in good agreement. A minor disagreement in the above results is only due to a difference in conversion between British and SI units. Since series (31) and (32) converge fast, only the first 10 terms were applied. The convergence is presented in Table 4.

Example 2

Figure 6 shows a case of a more general longitudinal shape, cf. Table 2, of the cantilever beam of linearly elastic and Ludwick type non-linearly elastic material.

The behavior of the beam is similar to that obtained for the case of the prismatic beam, Example 1, since the same combinations of material constants are used, Table 1. However, it is shown that the radius of curvature is no longer constant, i.e., deflection curves are no longer part of circular lines. The geometry of the beam in this particular example enables a rapid decrease of the radius of curvature towards the end of the cantilever beam. The pattern is apparent.

An interesting behavior can be found, e.g., in cases t_1 and t_6 when subjected to $M_e = 5.0\,\text{kNm}$. In the first segment of the beam the radius of curvature of t_1 is smaller than of t_6, but towards the end of the beam the situation is reversed. It can be concluded that in general it is difficult to predict the behavior of the beam only on the basis of material parameters. Therefore each case should be analyzed with care.

Example 3

Example 3 is a case of a non-prismatic cantilever beam, cf. Table 2, made from annealed copper with material parameters $E = 458.501\,\text{MPa}$, $k = 0.463$ shown in Figure 7.

Deflection modes obtained via numerical and analytical approaches are practically identical. Some numerical values from Figure 7 are presented in Table 5.

The convergence of series (31) and (32) for this case is presented in Table 6. As expected, the convergence is not so rapidly achieved as in the case discussed in Example 1 due to greater complexity of the shape of the beam. A required accuracy was obtained after employing 20 terms of the series.

CONCLUSION

The above analysis discusses large deflection behavior of non-prismatic cantilever beams of rectangular cross-section which are constructed of non-linearly elastic material of Ludwick type and subjected to bending moments on the free end. In addition, the influence of different moduli of elasticity in the tensile and compressive domains is examined. The solution of governing equations of bending is obtained numerically. It is shown that in the case when the non-linear stress–strain relationships in the tensile and compressive domains are identical, an analytical solution can be found in terms of infinite series. Good agreement between the numerical and analytical approaches have been confirmed by the results from the previously published studies.

In general, the results of the above analysis indicate that the deflection profile of the beam depends, besides the loading conditions, also on many other factors including the longitudinal shape of the beam and rheological model of the material. Moreover, the same analysis may be used for treating the problem of a general non-linear relation between stress and strain of the form $\sigma = f(\varepsilon)$, where f is an experimental stress–strain function characterizing the structural material.

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